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Corner Spontaneous Magnetization

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Details are given of a new method allowing an exact calculation of the spontaneous magnetization in the corner as well as along the edge at an arbitrary distance of the corner for a rectangular planar Ising ferromagnet.

KEY WORDS: Ising models; exact calculations; critical exponents; corner magnetization; magnetization profiles; inhomogeneous systems.

INTRODUCTION

In this paper, we give the details of an exact calculation, recently published,[9] of the spontaneous magnetization at points near the corner but on the edge of a rectangular Ising lattice of infinite extent. There has been considerable interest recently in the critical behavior of inhomogeneous systems, reviewed by Iglói *et al.*[1] For instance, in a wedge-shaped planar lattice with opening angle θ, conformal-theoretic methods, together with scaling, predict that the critical exponent for the apical spontaneous magnetization is $\beta_c(\theta)=\pi/2\theta$.[2,3] The important result $\beta_c(\pi)=1/2$ has been known for some time.[4] Capturing the result for $\theta=\pi/2$ has been considerably more troublesome. To start with, the usual Töplitz-determinantal methods[5] do not appear to apply in the corner, presumably because of the lower symmetry. To date, the result $\beta_c(\pi/2)=1$ has only been established by numerical solution of certain transfer matrix equations, usually (but not always) in the Hamiltonian limit.[3,6–8] But in defense of such methods, Kaiser and Peschel[8] conjectured what is now seen to be the correct

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functional form of the corner magnetization.[9] As a bonus, our method allows us to study the crossover between corner and edge magnetization analytically.

1. SETUP

Consider a rectangular Ising ferromagnetic lattice with spins $\sigma_{i,j} = \pm 1$ located at sites (i, j) with $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. A configuration denoted $\{\sigma\}$ of such spins on the lattice has energy

$$E_{N,M}(\{\sigma\}) = -J_1 \sum_{j=1}^{N-1} \sum_{i=1}^{M} \sigma_{i,j}\sigma_{i,j+1} - J_2 \sum_{j=1}^{N} \sum_{i=1}^{M-1} \sigma_{i,j}\sigma_{i+1,j} \tag{1.1}$$

where J_1 and J_2 are positive nearest neighbor couplings. The lattice is assumed to be in thermal equilibrium with a heat bath at inverse temperature β with a normalized configurational probability

$$P_{N,M}(\{\sigma\}) = Z_{N,M}^{-1} \exp[-\beta E_{N,M}(\{\sigma\})] \tag{1.2}$$

Here, we are interested in the spontaneous magnetization on the edge of such a lattice, at a distance n from the corner; this is given by

$$m_c^2(n) = \lim_{N,M \to \infty} \langle \sigma_{n,1}\sigma_{n,N} \rangle \tag{1.3}$$

where $\langle \cdot \rangle$ refers to the expectation value with respect to the probability distribution (1.2). Physically, this expression holds provided the thermodynamic limit is taken in such a way that with probability one asymptotically, we have a single magnetized domain in the system. We shall return to this point later [cf. discussion of (3.1)].

We use a transfer matrix working in the vertical direction. Let the matrix T_1 account for Boltzmann factors between rows of spins, i.e.,

$$T_1((\sigma_i), (\sigma_i')) = \exp\left(K_1 \sum_{i=1}^{M} \sigma_i \sigma_i'\right) \tag{1.4}$$

where $K_1 = \beta J_1$, and let T_2 account for the ones occurring within a row between spins on neighboring columns: T_2 is a diagonal matrix given by

$$T_2((\sigma_i), (\sigma_i')) = \mathbf{1}((\sigma_i), (\sigma_i')) \exp\left(K_2 \sum_{i=1}^{M-1} \sigma_i \sigma_{i+1}'\right) \tag{1.5}$$

where $K_2 = \beta J_2$. Then

$$\langle \sigma_{n,1} \sigma_{n,N} \rangle = \frac{\varphi^T \sigma_{n,N} (T_2 T_1)^{N-1} T_2 \sigma_{n,1} \varphi}{\varphi^T (T_2 T_1)^{N-1} T_2 \varphi} \tag{1.6}$$

where the vector φ assigns equal weight to every configuration of spins in a row, as required by the free boundary conditions.

We now introduce a Hilbert space H_M for a row of spins as the tensorial product of M copies of the spin-1/2 space denoted H, the jth of these spaces corresponding to the jth spin on the row, counting from the left: the spin operators σ_j^α, where $\alpha = x, y, z$, are defined by

$$\sigma_j^\alpha = \left(\bigotimes_1^{j-1} \mathbf{1} \right) \otimes \sigma^\alpha \otimes \left(\bigotimes_{j+1}^{M} \mathbf{1} \right)$$

where the σ^α are Pauli spin operators. We consider transfer operators V_1 and V_2 whose respective spin representatives in the representation with σ_j^x diagonal for $1 \leqslant j \leqslant M$ are the matrices $(2 \sinh 2K_1)^{-M/2} T_1$ and T_2. It is easily seen that these operators are self-adjoint and given by

$$V_1 = \exp\left(-K_1^* \sum_{j=1}^{M} \sigma_j^z \right), \qquad V_2 = \exp\left(K_2 \sum_{j=1}^{M-1} \sigma_j^x \sigma_{j+1}^x \right) \tag{1.7}$$

where the dual coupling constant K_1^* is defined by $\exp(-2K_1^*) = \tanh K_1$. Equation (1.6) becomes

$$\langle \sigma_{n,1} \sigma_{n,N} \rangle = \frac{\langle 0 | \, \sigma_n^x (V_2 V_1)^{N-1} V_2 \sigma_n^x \, | 0 \rangle}{\langle 0 | \, (V_2 V_1)^{N-1} V_2 \, | 0 \rangle} \tag{1.8}$$

where the state $|0\rangle$ is represented by the vector φ introduced in (1.6) to describe the free boundary conditions; this state can be defined by $\sigma_j^z |0\rangle = -|0\rangle$ for $1 \leqslant j \leqslant M$. The factor $(2 \sinh 2K_1)^{M/2}$ left out in the definition of V_1 gets canceled out between the numerator and denominator.

It is useful to introduce the symmetrized form $V' = V_1^{1/2} V_2 V_1^{1/2}$ for analysis of (1.8), giving

$$\langle \sigma_{n,1} \sigma_{n,N} \rangle = \frac{\langle 0 | \, \hat{\sigma}_n V'^N \hat{\sigma}_n^\dagger \, | 0 \rangle}{\langle 0 | \, V'^N \, | 0 \rangle} \tag{1.9}$$

where $\hat{\sigma}_n = V_1^{1/2} \sigma_n V_1^{-1/2}$. This will be developed in the next section by spectral decomposition of V'.

2. SPECTRUM OF THE TRANSFER MATRIX

Fortunately the spectrum of V' has been worked out elsewhere,[10, 11] so we shall introduce some notations and summarize the results. First, consider the Jordan–Wigner[12, 13] transformation to fermions defined by $f_j^\dagger = P_{j-1}\sigma_j^+$, where $\sigma_j^+ = (\sigma_j^x + i\sigma_j^y)/2$ is the spin raising operator, $P_0 = 1$, and $P_j = \prod_{k=1}^{j}(-\sigma_k^z)$ for $1 \leqslant j \leqslant M$. The anticommutators are the fermionic ones $[f_j^\dagger, f_k]_+ = \delta_{jk}$ and $[f_j, f_k]_+ = 0$, and the Γ_j defined by $\Gamma_{2j-1} = f_j^\dagger + f_j$ and $\Gamma_{2j} = -i(f_j^\dagger - f_j)$ satisfy

$$[\Gamma_j, \Gamma_k]_+ = 2\delta_{jk} \tag{2.1}$$

and are therefore spinors.[14] We express the transfer operators (1.7) in terms of the spinors using

$$\begin{aligned} \sigma_j^z &= -i\Gamma_{2j-1}\Gamma_{2j} && \text{for} \quad 1 \leqslant j \leqslant M \\ \sigma_j^x \sigma_{j+1}^x &= i\Gamma_{2j}\Gamma_{2j+1} && \text{for} \quad 1 \leqslant j \leqslant M-1 \end{aligned} \tag{2.2}$$

If the lattice was a cylinder rather than a rectangle, we would have to consider the additional term $\sigma_M^x \sigma_1^x = -iP_M \Gamma_{2M}\Gamma_1$. The action of the operator P_M, which we shall hereafter denote by P, is to flip all spins in the row; P is a symmetry of the problem:

$$[P, V_1]_- = 0, \qquad [P, V_2]_- = 0 \tag{2.3}$$

The quadratic structure of (2.2) in the Γ_j together with the anticommutation rules (2.1) implies a linear Euclidean evolution

$$V'\Gamma^T V'^{-1} = \Gamma^T R \tag{2.4}$$

The existence of V'^{-1} is clear for finite M. R is self-adjoint, and satisfies $RR^T = 1$ because its action on the spinors preserves their Clifford structure (2.1). It can be diagonalized and its eigenvalues occur in pairs $e^{\pm\gamma(k)}$, where $\gamma(k)$ is the nonnegative solution of

$$\cosh \gamma(k) = \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos k \tag{2.5}$$

If we define the eigenvector y_k by $Ry_k = e^{\gamma(k)} y_k$, then $Ry_k^* = e^{-\gamma(k)} y_k^*$. The operators

$$X_k = \sum_{j=1}^{2M} y_{jk} \Gamma_j \tag{2.6}$$

satisfy the evolution $V' X_k V'^{-1} = e^{\gamma(k)} X_k$. Provided that the eigenvectors are normalized by $\|y_k\|^2 = 1/2$, these operators are fermionic, allowing us to define new spinors

$$\hat{\Gamma}_{2k-1} = X_k^\dagger + X_k, \qquad \hat{\Gamma}_{2k} = -i(X_k^\dagger - X_k) \tag{2.7}$$

They are related to the old ones by a matrix ξ

$$\hat{\Gamma}_k = \sum_{j=1}^{2M} \xi_{jk} \Gamma_j \tag{2.8}$$

with all the elements of ξ being real and $\xi\xi^T = \xi^T\xi = 1$. This permits us to invert the relationship (2.8). Expressing ξ in term of the eigenvectors y gives

$$f_j^\dagger = \sum_k X_k^\dagger (y_{2j-1,k} + i y_{2j,k}) + X_k (y_{2j-1,k}^* + i y_{2j,k}^*) \tag{2.9}$$

Since V' is unimodular, the diagonalization of R translates into

$$V' = \exp\left(-\sum_k \gamma(k)(X_k^\dagger X_k - \tfrac{1}{2}) \right) \tag{2.10}$$

There are $2M$ eigenvectors, hence M values of k giving nontrivially different y_k; these are the solutions of[10]

$$e^{iMk} = -i\alpha e^{i\delta^*(k)} \tag{2.11}$$

where $\alpha = \pm i$ and δ^* can be defined by

$$e^{2i\delta^*(k)} = \frac{(Be^{ik} - 1)(e^{ik} - A)}{(Ae^{ik} - 1)(e^{ik} - B)} \qquad \text{and} \qquad e^{i\delta^*(0)} = 1 \tag{2.12}$$

with $A = \coth K_1^* \coth K_2$ and $B = \coth K_1^* \tanh K_2$.

The appearance of α can be understood by appealing to the reflection symmetry Σ possessed by V', which is defined by $\Sigma \sigma_j^q \Sigma^{-1} = \sigma_{M+1-j}^q$ for $q = x, y, z$ and $1 \leqslant j \leqslant M$. This imples

$$\begin{aligned} \Sigma \Gamma_{2j-1} \Sigma^{-1} &= -iP\Gamma_{2M-2j+2} \\ \Sigma \Gamma_{2j} \Sigma^{-1} &= iP\Gamma_{2M-2j+1} \end{aligned} \tag{2.13}$$

Since the eigenvectors are given by

$$y_{2j,k} = N_k \sin(kj - \varphi_0(k)) \qquad \text{and} \qquad y_{2j-1,k} = iN_k \sin(kj - \varphi_1(k)) \tag{2.14}$$

where

$$
\begin{aligned}
e^{2i\varphi_0(k)} &= \frac{A - e^{ik}}{A - e^{-ik}} \qquad \text{and} \qquad e^{i\varphi_0(0)} = 1 \\
e^{2i\varphi_1(k)} &= \frac{1 - Be^{ik}}{1 - Be^{-ik}} \qquad \text{and} \qquad e^{i\varphi_1(0)} = \operatorname{sgn}(T - T_c)
\end{aligned}
\tag{2.15}
$$

then (2.11) gives

$$
y_{2M-2j+2} = -\alpha y_{2j-1}, \qquad y_{2M-2j+1} = \alpha y_{2j} \tag{2.16}
$$

This relationship, which was deduced from the structure of R, implies an interesting reflection behavior, namely $\Sigma X_k \Sigma^{-1} = -i\alpha P X_k$. Using (2.16) in (2.9) gives

$$
\begin{aligned}
f_j^\dagger \pm f_{M-j+1}^\dagger = \sum_k \big(& X_k^\dagger (1 \mp i\alpha_k)(y_{2j-1,k} + iy_{2j,k}) \\
& + X_k (1 \pm i\alpha_k)(y_{2j-1,k}^* + iy_{2j,k}^*)\big)
\end{aligned}
\tag{2.17}
$$

Returning to (2.11), it turns out that below the critical temperature, and when M is big enough, there are only $M-1$ solutions with k real giving distinct eigenvectors. The last mode has a pure imaginary wavevector $k = i\hat{\gamma}(0) + O(\exp[-2M\hat{\gamma}(0)])$, where the function $\hat{\gamma}$ is defined by (2.5) with K_1 and K_2 interchanged; we have $\exp[\hat{\gamma}(0)] = B$. This mode gives asymptotic degeneracy in the spectrum since $\gamma(i\hat{\gamma}(0)) = 0$; precisely,

$$
\gamma_c = \gamma(k) = 2 \sinh 2K_1^* \sinh \hat{\gamma}(0)\, e^{-M\hat{\gamma}(0)}(1 + O(e^{-M\hat{\gamma}(0)})) \tag{2.18}
$$

The expression for the corresponding eigenvector is

$$
\begin{aligned}
y_{2j,c} &= \left[\frac{\sinh \hat{\gamma}(0)}{2}\right]^{1/2} e^{-(M-j+1/2)\hat{\gamma}(0)} + O(e^{-M\hat{\gamma}(0)}) \\
y_{2j-1,c} &= i\left[\frac{\sinh \hat{\gamma}(0)}{2}\right]^{1/2} e^{-(j-1/2)\hat{\gamma}(0)} + O(e^{-M\hat{\gamma}(0)})
\end{aligned}
\tag{2.19}
$$

and its reflection behavior is given by $\alpha_c = i$.

The final detail needed to develop (1.9) is the vacuum for the X_k, denoted $|\Phi\rangle$. This state is the maximum eigenvector of V', is nondegenerate for any temperature (provided M is finite), and since V' is invariant under parity, $|\Phi\rangle$ is an eigenstate of P; the corresponding eigenvalue is conserved by continuity, so we can determine it by taking $T \to \infty$, in which case V' goes to V_1, and $|\Phi\rangle$ goes to $|0\rangle$ (a row of free spins) which is even, so at any temperature $P\,|\Phi\rangle = |\Phi\rangle$.

3. DERIVATION OF AN INTEGRAL EQUATION FOR THE MATRIX ELEMENT

Returning to (1.9), notice that $\gamma(k) \geqslant \gamma(0) > 0$ strictly away from the critical point. Thus we only expect the term

$$e^{N\gamma_c} \left| \frac{\langle 0 | \hat{\sigma}_j X_c^\dagger | \Phi \rangle}{\langle 0 | \Phi \rangle} \right|^2 \tag{3.1}$$

to be significant in the spectral decomposition, and then only if the thermodynamic limit is taken so that $N\gamma_c \to 0$ as $M, N \to \infty$. Recalling (2.18), this means N cannot grow faster than $\exp[M\hat{\gamma}(0)]$. This limit on the acicularity of the domain suppresses the formation of more than one magnetic domain, just as it did in the scaling theory of surface tension on a cylinder.[15, 17] Thus we expect the magnetization along the lower face of the strip at a distance j from the corner to be given by

$$m_c(j) = e^{K_1^*} \lim_{M \to \infty} \left| \frac{\langle 0 | f_j X_c^\dagger | \Phi \rangle}{\langle 0 | \Phi \rangle} \right| \tag{3.2}$$

This is obvious for $j = 1$ and follows for $2 \leqslant j \leqslant M$ because

$$\langle 0 | \hat{\sigma}_j = e^{K_1^*} \langle 0 | \prod_{l=1}^{j-1} (-\sigma_l^z) f_j \tag{3.3}$$

and $\langle 0 | (-\sigma_l^z) = \langle 0 |$. Such a reduction to a bilinear form as appears in (3.2) is special to the edge. If we go to the next layer in, the analogous equation involving four Fermi operators can be given, and so on.

Using reflection symmetry and (2.17) gives

$$m_c(j) = \frac{e^{K_1^*}}{2} \lim_{M \to \infty} \left((1 - i\alpha_c)(y_{2j-1,c}^* - i y_{2j,c}^*) \right.$$

$$\left. - \sum_{k \in (0,\pi)} (1 + i\alpha_k)(y_{2j-1,k} - i y_{2j,k}) \frac{\langle 0 | X_k^\dagger X_c^\dagger | \Phi \rangle}{\langle 0 | \Phi \rangle} \right) \tag{3.4}$$

Using oddness in k of $y_{j,k}$ and $X_k^\dagger$, we have

$$m_c(j) = e^{K_1^*} \lim_{M \to \infty} \left((y_{2j-1,c}^* - i y_{2j,c}^*) - \sum_{\substack{k \in (-\pi,\pi) \\ \alpha_k = -i}} N_k^2 e^{ikj} \frac{e^{-i\varphi_0(k)} - e^{-i\varphi_1(k)}}{e^{-i\varphi_0(k)} + e^{-i\varphi_1(k)}} K_M(k) \right) \tag{3.5}$$

where

$$K_M(k) = \frac{e^{-i\varphi_0(k)} + e^{-i\varphi_1(k)}}{N_k} \frac{\langle 0|\, X_k^\dagger X_c^\dagger \,|\Phi\rangle}{\langle 0 \mid \Phi\rangle} \tag{3.6}$$

for $\alpha_k = -i$. The problem of evaluating this matrix element can be solved from the $|0\rangle$ vacuum property by considering $\langle 0|\, f_j^\dagger X_c^\dagger \,|\Phi\rangle = 0$ for $1 \leqslant j \leqslant M$. Using reflection symmetry and (2.17) with a plus sign, then oddness and (3.6) give

$$\frac{1}{2} \sum_{\substack{k \in (-\pi, \pi) \\ \alpha_k = -i}} N_k^2 e^{ikj} K_M(k) = i \frac{(B^2 - 1)^{1/2}}{2} \left(e^{-j\hat{\gamma}(0)} - e^{-(M-j+1)\hat{\gamma}(0)}\right) \tag{3.7}$$

Choosing the minus sign in (2.17) leads to a similar equation for $\alpha_k = i$, but with zero on the right-hand side, which strongly suggests a selection rule associated with the reflection symmetry.

We now convert (3.7) to an integral equation. We first multiply by e^{-ijq} and sum on j. We consider only wavevectors q satisfying a constraint similar to (2.11) for k, so that the appearance of terms with $k = q$ is under control for any M. Equation (3.7) becomes

$$\begin{aligned} &\sum_{\substack{k \in (-\pi, \pi) \\ \alpha_k = -i}} \frac{e^{i(k-q)}}{1 - e^{i(k-q)}} \left(1 + e^{i(\delta^*(k) - \delta^*(q))}\right) N_k^2 K_M(k) \\ &\quad = i(B^2 - 1)^{1/2} \left(\frac{B^{-1}}{e^{iq} - B^{-1}} + \frac{e^{-i\delta^*(q)}}{e^{iq} - B}\right) \end{aligned} \tag{3.8}$$

for $\alpha_q = i$, and

$$\begin{aligned} &M N_q^2 K(q) + \sum_{\substack{k \in (-\pi, \pi) \\ \alpha_k = -i \\ k \neq q}} \frac{e^{i(k-q)}}{1 - e^{i(k-q)}} \left(1 + e^{i(\delta^*(k) - \delta^*(q))}\right) N_k^2 K_M(k) \\ &\quad = i(B^2 - 1)^{1/2} \left(\frac{B^{-1}}{e^{iq} - B^{-1}} - \frac{e^{-i\delta^*(q)}}{e^{iq} - B}\right) \end{aligned} \tag{3.9}$$

for $\alpha_q = -i$.

We give in Appendix A an informal argument showing that the sequence $K_M(e^{ik})$ has a subsequence converging uniformly in an annulus

$\alpha^{-1} < |e^{ik}| < \alpha$, allowing us to take the thermodynamic limit on (3.8) and (3.9), thereby giving integral equations for the limiting K:

$$\frac{P}{2\pi}\int_{-\pi}^{\pi} dk \frac{e^{i(k-q)}}{e^{i(k-q)}-1}(1+e^{i(\delta^*(k)-\delta^*(q))})K(k)$$

$$= -2i(B^2-1)^{1/2}\left(\frac{B^{-1}}{e^{iq}-B^{-1}}+\frac{e^{-i\delta^*-i\delta^*(q)}}{e^{iq}-B}\right) \tag{3.10}$$

$$-K(q)+\frac{P}{2\pi}\int_{-\pi}^{\pi} dk \frac{e^{i(k-q)}}{e^{i(k-q)}-1}(1-e^{i(\delta^*(k)-\delta^*(q))})K(k)$$

$$= -2i(B^2-1)^{1/2}\left(\frac{B^{-1}}{e^{iq}-B^{-1}}-\frac{e^{-i\delta^*(q)}}{e^{iq}-B}\right) \tag{3.11}$$

Summing these two equations, we obtain

$$K(q)-\frac{P}{\pi}\int_{-\pi}^{\pi} dk \frac{e^{i(k-q)}}{e^{i(k-q)}-1}K(k) = 4i(1-B^{-2})^{1/2}\frac{1}{e^{iq}-B^{-1}} \tag{3.12}$$

which we shall solve in the next section.

4. SOLUTION OF THE SINGULAR INTEGRAL EQUATION

The Hilbert transform $\mathbf{H}$ is defined for functions analytic on and near the unit circle by the principal part integral

$$(\mathbf{H}f)(z) = \frac{P}{i\pi}\int_{|t|=1}\frac{t}{t-z}f(t) \tag{4.1}$$

We set $z = e^{ik}$ and introduce $K(z)$ as a synonym for $K(k)$; the function $K(z)$ is expected to be analytic in an annulus $\alpha^{-1} < |z| < \alpha$. The problem posed by (3.12) is to solve

$$K(z) - (\mathbf{H}K)(z) = \frac{4i(1-B^{-2})^{1/2}}{z-B^{-1}} \tag{4.2}$$

This means that the function

$$K_+(z) = K(z) - \frac{2i(1-B^{-2})^{1/2}}{z-B^{-1}} \tag{4.3}$$

satisfies $(1-\mathbf{H})K_+=0$. The function K_+ is analytic in the same annulus $\alpha^{-1}<|z|<\alpha$ as K provided $\alpha<B$, hence can be developed in Laurent series, and (4.3) says that terms z^n with $n<0$ vanish in this expansion, or equivalently that K_+ is analytic for $|z|<\alpha$. Its form is determined from the oddness in k of $y_{m,k}$, which gives $K(z^{-1})=\{\exp[i(k+\delta^*(k))]\}\,K(z)$, hence a relationship between $K_+(z^{-1})$ and $K_+(z)$. First note that $\exp[i\delta^*(k)]=\tau(e^{ik})\,\tau^{-1}(e^{-ik})$, where τ is defined by

$$\tau(z)=\left(\frac{z-A}{z-B}\right)^{1/2} \quad \text{and} \quad \tau(1)>0 \tag{4.4}$$

Then we have

$$z\tau(z)\,K_+(z)-\tau(z^{-1})\,K_+(z^{-1})=-2i(1-B^{-2})^{1/2}\left(\frac{z\tau(z)}{z-B^{-1}}+\frac{Bz\tau(z^{-1})}{z-B}\right) \tag{4.5}$$

The first term of the left-hand side of this equation is analytic for $|z|<\alpha$, whereas the second term is analytic for $|z|>\alpha^{-1}$, and the right-hand side is analytic on the annulus, allowing application of the Wiener–Hopf technique; the operators $\frac{1}{2}(1+\mathbf{H})$ and $\frac{1}{2}(1-\mathbf{H})$, respectively, project on the spaces of functions "analytic inside" and "analytic outside, vanishing at infinity."

We find that

$$z\tau(z)\,K_+(z)=2i(1-B^{-2})^{1/2}\left(\frac{z\tau(z)}{z-B^{-1}}-\frac{B^{-1}\tau(B^{-1})}{z-B^{-1}}\right.$$
$$\left.+\frac{B^2\tau(B^{-1})}{z-B}+B\tau(0)\right) \tag{4.6}$$

is an entire function (analytic everywhere), bounded by a constant, since its Laurent series has no terms with $n\geqslant 0$, so by Liouville's theorem is a constant; calculating its value for $z=0$ finally gives the result

$$K(z)=2i(1-B^{-2})^{1/2}\,\frac{\tau(B^{-1})(1-B)(1+z)\,\tau^{-1}(z)}{(z-B)(z-B^{-1})} \tag{4.7}$$

or equivalently without using τ

$$K(z)=-4i(A-B^{-1})^{1/2}\,\frac{\sinh(\hat{\gamma}(0)/2)(1+z)}{[(z-A)(z-B)]^{1/2}(z-B^{-1})} \tag{4.8}$$

provided the square root in the denominator is chosen negative for $z=1$. We have proved that this is the only solution to (3.12), so that all uniformly convergent subsequences of K_M (see Appendix A) have this same limit, proving that the sequence itself converges uniformly to K.

Equation (3.10) could be solved directly by noting that the operator on the left-hand side first occurred in Yang's work on the bulk spontaneous magnetization[18] (if we replace B by B^{-1}). Thus we extend Yang's analysis to the case $T > T_c$.[19,20] The integral operator is invertible explicitly using Onsager's elliptic substitution[25] (see Appendix B). This analysis, which is the first method by which we obtained the result, is considerably more protracted than the Wiener–Hopf method described above.

5. MAGNETIZATION

We take the thermodynamic limit of (3.5) as we did for (3.10):

$$m_e(j) = \left| \frac{e^{K_1^*}}{2} \left(-i(B^2-1)^{1/2} B^{-j} \right.\right.$$
$$\left.\left. - \frac{P}{4i\pi} \int_{|z|=1} z^{j-1}\, dz\, \frac{e^{-i\varphi_0(k)} - e^{-\varphi_1(k)}}{e^{-i\varphi_0(k)} + e^{-i\varphi_1(k)}} K(z) \right) \right| \tag{5.1}$$

Using

$$\frac{e^{-i\varphi_0(k)} - e^{-\varphi_1(k)}}{e^{-i\varphi_0(k)} + e^{-\varphi_1(k)}} = \frac{e^{-2i\varphi_0(k)} + e^{-2i\varphi_1(k)}}{e^{-2i\varphi_0(k)} - e^{-2i\varphi_1(k)}} - \frac{2e^{-\varphi_0(k) - i\varphi_1(k)}}{e^{-2i\varphi_0(k)} - e^{-2i\varphi_1(k)}} \tag{5.2}$$

allows us to decompose the integrand in (5.1) into two terms with different analytic structures. The first term

$$z^{j-1} \frac{e^{-2i\varphi_0(k)} + e^{-2i\varphi_1(k)}}{e^{-2i\varphi_0(k)} - e^{-2i\varphi_1(k)}} K(z) \tag{5.3}$$

has a branch cut between A and B and a pole in B^{-1} due to K, a pole in 1 from to the other factor, and a singularity at infinity because of z^{j-1}; its principal part integral is thus

$$P \int z^{j-1} \frac{e^{-2i\varphi_0(k)} + e^{-2i\varphi_1(k)}}{e^{-2i\varphi_0(k)} - e^{-2i\varphi_1(k)}} K(z)\, dz$$
$$= -4\pi \left[\frac{(A-1)(B-1)}{AB-1} \right]^{1/2} + 4\pi B^{-j} (B^2-1)^{1/2} \tag{5.4}$$

The second term on the right-hand side (residue at B^{-1}) cancels exactly the contribution of the $X_c^\dagger$ mode in (5.1).

The second term arising from the decomposition (5.2)

$$z^{j-1}\left(-\frac{2e^{-i\varphi_0(k)-i\varphi_1(k)}}{e^{-2i\varphi_0(k)}-e^{-2i\varphi_1(k)}}\right)K(z) \tag{5.5}$$

has a branch cut between A^{-1} and B^{-1}, a pole in 1, and a singularity at infinity. For $j=1$ that singularity is a single pole and we obtain

$$P\int z^{j-1}\left(-\frac{2e^{-i\varphi_0(k)-i\varphi_1(k)}}{e^{-2i\varphi_0(k)}-e^{-2i\varphi_1(k)}}\right)K(z)\,dz$$
$$=4\pi\left[\frac{(A-1)(B-1)}{AB-1}\right]^{1/2}-16\pi\frac{\sinh(\hat{\gamma}(0)/2)}{(A-B^{-1})^{1/2}} \tag{5.6}$$

Hence the corner magnetization is

$$m_e(1)=e^{K_1^*}\frac{1-B^{-1}}{(1-A^{-1}B^{-1})^{1/2}}=\frac{B^{1/2}-B^{-1/2}}{B^{1/2}-A^{-1/2}} \tag{5.7}$$

The critical exponent is immediately seen to be 1, since $\hat{\gamma}(0)=\ln B\propto(T_c-T)/T_c$ near the critical temperature. Equation (5.7) can be turned into a formula conjectured by C. Kaiser and I. Peschel where symmetry in the exchange of K_1 and K_2 is evident:

$$m_e(1)=1-\tfrac{1}{2}(\coth K_1-1)(\coth K_2-1) \tag{5.8}$$

For large j it is not longer practical to calculate the residue at infinity of (5.5) and we express (5.6) in term of an integral along the branch cut

$$P\int z^{j-1}\left(-\frac{2e^{-i\varphi_0(k)-i\varphi_1(k)}}{e^{-2i\varphi_0(k)}-e^{-2i\varphi_1(k)}}\right)K(z)\,dz$$
$$=-4\pi\left[\frac{(A-1)(B-1)}{AB-1}\right]^{1/2}$$
$$+8\frac{\sinh(\hat{\gamma}(0)/2)}{(B-A^{-1})^{1/2}}\int_{x=A^{-1}}^{B^{-1}}\frac{x^{j-1}\,dx}{1-x}\left(\frac{x-A^{-1}}{1-Bx}\right)^{1/2} \tag{5.9}$$

giving

$$\frac{m_e(j)}{m_e}=1-\frac{1}{\pi}\left(\frac{1-B^{-1}}{1-A^{-1}}\right)^{1/2}\int_{x=A^{-1}}^{B^{-1}}\frac{x^{j-1}\,dx}{1-x}\left(\frac{x-A^{-1}}{B^{-1}-x}\right)^{1/2} \tag{5.10}$$

where the edge magnetization m_e is

$$m_e = e^{K_1^*}\left(\frac{(A-1)(B-1)}{AB-1}\right)^{1/2} = \left(\frac{\coth 2K_1^* - \coth 2K_2}{\coth 2K_1^* - 1}\right)^{1/2} \tag{5.11}$$

in accordance with ref. 4. We consider two limits of Eq. (5.10). First, for $j \to \infty$ at constant temperature the integral is dominated by the neighborhood of B^{-1}; using a change of integration variable to $x = B^{-1}e^{-t}$ gives

$$\frac{m_e(j)}{m_e} \approx 1 - B^{-j}\left(\frac{1 - A^{-1}B}{(1-A^{-1})(1-B^{-1})}\right)^{1/2} \frac{1}{\pi}\int_{t=0}^{\ln(A/B)} \frac{e^{-jt}}{\sqrt{t}}\,dt \tag{5.12}$$

i.e.,

$$m_e(j) \approx m_e - e^{K_1^*} e^{-(j-1)\hat{\gamma}(0)}(\pi j \sinh 2K_1 \sinh 2K_2)^{-1/2}$$

A short discussion elsewhere[9] makes it physically reasonable that the prefactor in the exponential decay of (5.12) is $j^{-1/2}$ by appealing to the bubble model.[21]

The second limit is the scaling one, which is taken approaching the critical point with the distance j increasing proportionally to the correlation length; we choose as rescaled distance the parameter $r = j\hat{\gamma}(0)$. Using the auxiliary variable u defined by $x = \exp[-\hat{\gamma}(0)(1+u^2)]$ gives the following expression for the scaling function:

$$F(r) = \lim_{\substack{\hat{\gamma}(0)\to 0\\ j\hat{\gamma}(0) = r}} \frac{m_e(j)}{m_e} = 1 - \frac{2}{\pi}\int_{u=0}^{\infty} \frac{e^{-r(1+u^2)}}{1+u^2}\,du \tag{5.13}$$

Differentiating and then reintegrating with respect to r gives F as a Gaussian error function:

$$F(x) = \int_{y=0}^{x} \frac{e^{-y}\,dy}{(\pi y)^{1/2}} \frac{2}{\sqrt{\pi}}\int_{y=0}^{\sqrt{x}} e^{-y^2}\,dy = \operatorname{erf}(\sqrt{x}) \tag{5.14}$$

The short-distance behavior is $F(x) \approx 2(x/\pi)^{1/2}$. Note that using this formula and m_e to obtain the corner magnetization gives the correct exponent (one), but not the correct prefactor.

APPENDIX A. AN INFORMAL CONVERGENCE ARGUMENT

The matrix element

$$\frac{\langle 0|\, X_k^\dagger X_c^\dagger \,|\Phi\rangle}{N_k\langle 0 \mid \Phi\rangle} \tag{A.1}$$

can be expanded using (2.6) as a sum of the form

$$\sum_{j=1}^{M} e^{ijk} \frac{\langle 0|\, f_j X_c^\dagger \,|\Phi\rangle}{\langle 0 \mid \Phi\rangle} \tag{A.2}$$

multiplied by functions independent of j analytic in the annulus $B^{-1} < |e^{ik}| < B$. We consider

$$\begin{aligned}(e^{ik}-1) \sum_{j=1}^{M} e^{ijk} \frac{\langle 0|\, f_i X_c^\dagger \,|\Phi\rangle}{\langle 0 \mid \Phi\rangle} \\ = \sum_{j=1}^{M} e^{ijk} \frac{\langle 0|\, (f_{j-1}-f_j)\, X_c^\dagger \,|\Phi\rangle}{\langle 0 \mid \Phi\rangle} + \text{surface terms}\end{aligned} \tag{A.3}$$

We anticipate (a spectral gap argument suggests this strongly, but we have no proof) that the magnetization along the edge decays exponentially away from the corner to the edge value on a length scale of the bulk correlation length, hence that

$$\left| \frac{\langle 0|\, (f_{j-1}-f_j)\, X_c^\dagger \,|\Phi\rangle}{\langle 0 \mid \Phi\rangle} \right| \leqslant C\alpha^{-j} \tag{A.4}$$

where C and α are independent of M, and $\alpha > 1$. Then each function in the sequence

$$(e^{ik}-1) \frac{\langle 0|\, X_k^\dagger X_c^\dagger \,|\Phi\rangle}{N_k \langle 0 \mid \Phi\rangle} \tag{A.5}$$

will be analytic and uniformly bounded on the annulus $\alpha^{-1} < |e^{ik}| < \alpha$ (provided $\alpha < B$). Thus, by a theorem[22] sometimes known as Montel's (Theorem 12.8a in ref. 23 is attributed to Montel) this sequence will contain at least one uniformly convergent subsequence, with an analytic function for limit. Similarly, the $K_M(e^{ik})$ will contain such a subsequence because a potential singularity at $e^{ik} = 1$ is removable.

Care is needed in taking the limit as $M \to \infty$ in (3.5), (3.8), and (3.9) since the k's which solve (2.11) are not uniformly distributed in general, so Cauchy principal part integrals could create problems.

Define

$$F_\alpha(k) = e^{iMk} + i\alpha e^{i\delta^*(k)} \tag{A.6}$$

Then the zeros of $F_\alpha(k)$ are the eigenvalues to be inserted in (2.14) and a tedious calculation shows that

$$|N_k|^{-2} = 2(M - \delta^{*(1)}(k)) = -2ie^{-iMk}\frac{dF_\alpha}{dk} \tag{A.7}$$

Let the function $f(k)$ be 2π-periodic and analytic in a strip $|\mathrm{Im}\, k| < p$. Consider a contour C which surrounds the zeros of $F_\alpha(k)$ which lie on the real axis in $-\pi < k \leqslant \pi$ but lie inside the intersection of $|\mathrm{Im}\, k| < p$ and the domain of analyticity of $\exp(i\delta^*)$. The residue theorem gives

$$\frac{1}{2i\pi}\oint_C dk\,\frac{e^{iMk}}{F_\alpha(k)} f(k) = \sum_{\substack{k \in (-\pi,\, \pi) \\ \alpha_k = \alpha}} \frac{f(k)\, e^{iMk}}{dF_\alpha/dk} = -2i \sum_{\substack{k \in (-\pi,\, \pi) \\ \alpha_i = \alpha}} |N_k|^2 f(k) \tag{A.8}$$

On the other hand, it is easily shown that

$$\lim_{M \to \infty} \int_C dk\,\frac{e^{iMk}}{F_\alpha(k)} f(k) = \int_{k=-\pi}^{\pi} f(k)\, dk \tag{A.9}$$

This is a Riemann integration result in partial disguise.

The cases of significance have singular $f(k)$. To obtain (5.1) from (3.5), note that the function standing for $f(k)$ has a simple pole at $k = 0$ (mod 2π). This value is not in the eigenvalue set, since $\alpha = -i$. Equation (A.8) is now

$$-2i \sum |N_k|^2 f(k) + \frac{1}{2}\,\mathrm{Res}\, f(k=0) = \frac{1}{2i\pi}\oint_C dk\,\frac{e^{iMk}}{F_{-i}(k)} f(k) \tag{A.10}$$

As $M \to \infty$, the right-hand side of (A.10) tends to

$$\frac{1}{2i\pi}\int_{-\pi - i\varepsilon}^{\pi - i\varepsilon} f(k)\, dk = \frac{P}{2i\pi}\int_{-\pi}^{\pi} f(k)\, dk + \frac{1}{2}\,\mathrm{Res}\, f(k=0) \tag{A.11}$$

Thus

$$\lim_{M \to \infty} 2 \sum |N_k|^2 f(k) = \frac{P}{2i\pi}\int_{-\pi}^{\pi} f(k)\, dk \tag{A.12}$$

a result which can be applied to both (3.5) and (3.8), noting that the extra singularity at $k = q$ is not in the solution set for k since $\alpha_k = -\alpha_q$. Finally, the singularity in the integrand of (3.9) is removable, so the strip-analytic case for f suffices.

APPENDIX B. SOLUTION OF THE INTEGRAL EQUATION BY ELLIPTIC SUBSTITUTION

We discuss the solution of (3.10) by an alternative method which goes back to the work of Yang[18] on the bulk spontaneous magnetization of the planar Ising ferromagnet. There, and in subsequent work by Chang[4] and Abraham,[19] the following singular integral operator is considered:

$$(Y_+ f)(q) = \frac{P}{\pi} \int_0^{2\pi} dk \frac{1}{e^{i(k-q)} - 1} [1 + e^{i(\delta'(k) - \delta'(q))}] f(k) \qquad \text{(B.1)}$$

where the angle δ' is obtained from the δ^* in this paper by replacing B with B^{-1}, with the branch determined by $\delta'(0) = \pi$.

Let us define the functions $f_0(k)$ and $L(k)$ by

$$f_0(k) = e^{ik}/[(e^{ik} - A)(e^{ik} - B^{-1})]^{1/2} \qquad \text{(B.2)}$$

and

$$e^{ik} K(k) = L(k) f_0(k) \qquad \text{(B.3)}$$

so that (3.10) becomes

$$\frac{P}{\pi} \int_{-\pi}^{\pi} dk \frac{1}{e^{i(k-q)} - 1} [1 + e^{i(\delta^*(k) - \delta^*(q))}] \frac{f_0(k)}{f_0(q)} L(k)$$
$$= -4i(B^2 - 1)^{1/2} B^{-1/2} \left(B^{-1/2} \left(\frac{e^{iq} - A}{e^{iq} - B^{-1}} \right)^{1/2} + A^{1/2} \left(\frac{e^{iq} - A^{-1}}{e^{iq} - B} \right)^{1/2} \right) \qquad \text{(B.4)}$$

where the right-hand side vanishes at $q = 0$. The point of this is that the new singular integral operator on L becomes particularly simple in action under the conformal transformation

$$z(u) = e^{i\omega} = \frac{k \operatorname{cn} ia \operatorname{dn} u - \operatorname{dn} ia \operatorname{cn} u + u(1 - k^2) \operatorname{sn} u}{M(u)} \qquad \text{(B.5)}$$

with

$$M(u) = \operatorname{dn} ia \operatorname{dn} u - k \operatorname{cn} ia \operatorname{cn} u \qquad \text{(B.6)}$$

which was introduced by Onsager[25] in terms of Jacobi elliptic functions.[26] Equations (B.5) and (B.6) are correct for all T, but the assignment of ia and k depends on $\operatorname{sgn}(T - T_c)$. For $T < T_c$ it is

$$k = (\sinh\ 2K_1 \sinh 2K_2)^{-1} \qquad \text{(B.7)}$$

and

$$\operatorname{sn} ia = i \sinh 2K_2 \tag{B.8}$$

It follows from (B.5), (B.6), and (2.5) that

$$\frac{d\omega}{du} = -\frac{(1-k^2)}{M(u)} \tag{B.9}$$

and that

$$\sinh \gamma = -i \operatorname{sn}\ ia \frac{1-k^2}{M(u)} \tag{B.10}$$

Our strategy for solving (B.4) is to determine the spectrum of the integral operator on the left-hand side and then to expand both $L(k(u))$ and the right-hand side of (B.4) in terms of the eigenvectors. Provided there is no zero eigenvalue, this procedure allows us to evaluate the coefficients of L in this expansion, and eventually to identify L. Calling $J(u, u')$ the kernel in the left-hand side of (B.4) expressed in terms of the u variables, we have the eigenvalue problem

$$P\int_0^{4K} J(u, u')\, g(u)\, du = \lambda g(u') \tag{B.11}$$

The following properties of $z(u)$ are useful:

1. $z(u)$ is double-periodic with periods $4K$ and $4iK'$.

2. $z(u)$ is singular at the simple zeros of $M(u)$, which are at $i(a+K')$ and $i(a+3K')$ modulo $4K$ and $4iK'$.

3. In general, there are two u values in the unit cell for each z value. For $T < T_c$ the exceptions are $z(iK') = A$, $z(3iK') = A^{-1}$, $z(2K+iK') = B$, and $z(2K+3iK') = B^{-1}$. If $T > T_c$, then B and B^{-1} are interchanged.

4. For z values where there are two u values, such u values give equal and opposite values of $\exp(i\delta^*)$. Thus the period rectangle in the u plane corresponds to both sheets of the Riemann surface of $\exp(i\delta^*)$ and its branch points become simple poles and zeros in the u plane.

These properties indicate that for u and u' in the same period rectangle, $J(u, u')$ has a simple pole at $u = u'$ with residue 2 and no other singularities.

To complete an investigation of (B.11), the periodicity of $J(u, u')$ is needed. Using the properties of the elliptic functions, we find that all factors in $J(u, u')$ except $f_0(k(u))$ and $f_0(k(u'))$ are double-periodic with periods $4K$ and $4iK'$.

The function $z(u)$ sends the line $0 \leqslant u < 4K$ into the unit circle in the z plane. Thus

$$\begin{aligned}\int_0^{2K} du \frac{d \log f_0}{du} &= -\oint dz \frac{d \log f_0}{dz} \\ &= -\frac{1}{2}\oint dz \left(\frac{1}{z-A} + \frac{1}{z-B^{-1}}\right) \\ &= \begin{cases} -i\pi & \text{for} \quad T < T_c \\ 0 & \text{for} \quad T > T_c \end{cases}\end{aligned} \tag{B.12}$$

Since $d \log f_0/du$ is periodic in u with period $4K$, (B.12) implies that for $T < T_c$, f_0 changes sign by translation of $4K$, and that for $T > T_c$, f_0 is periodic with period $4K$. On the other hand, the line $2K + iu$ with $0 \leqslant u \leqslant 4K'$ except for infinitesimal indentations at $2K + iK'$ and $2K + 3iK'$ goes into a union of two loops about $z = B$ and $z = B^{-1}$ connected by real lines of intermediate points. Thus a repeat of the argument involving (B.12) with the new contour shows that f_0 is always antiperiodic for shifts of $4iK'$.

$J(u, u')$ has the same periodicity properties in each of its variables, implying for $T < T_c$ that $g(u + 4K) = -g(u)$. It is clear that

$$g_m(u) = (4K)^{-1/2}\, e^{i(m+1/2)\pi u/2K} \tag{B.13}$$

are orthonormal eigenvectors of (B.11) with eigenvalues

$$\lambda_m = 2\left(\frac{1 - q^{2m+1}}{1 + q^{2m+1}}\right) \tag{B.14}$$

where m is any rational integer and $q = \exp(-\pi K'/K)$.

Continuing our strategy, we now check that the right-hand side of (B.4) is antiperiodic for translations in u of $4K$ and $4iK'$. Obtaining the expansion over the eigenvectors is relatively simple because the branch points become simple zeros and poles. The right-hand side of (B.4) is

$$\frac{-4i(AB-1)^{1/2}\pi}{B^{1/2}k^{1/2}K} \sum_{-\infty}^{\infty} (-1)^m e^{i(m+1/2)\pi u/2K} \frac{q^{(m+1/2)/2} - q^{3(m+1/2)/2}}{1 + q^{2m+1}} \tag{B.15}$$

so we obtain the result

$$L(k(u)) = \frac{-2i(AB-1)^{1/2}\pi}{B^{1/2}k^{1/2}K} \sum_{-\infty}^{\infty} (-1)^m e^{i(m+1/2)\pi u/2K} \frac{q^{(m+1/2)/2}}{1+q^{m+1/2}} \tag{B.16}$$

We now use a Landen transform with $q_0 = \sqrt{q}$ and the associated complete elliptic integrals $K_0 = (1+k)K$ and $K_0' = (1+k)K'/2$ with new modulus $k_0 = 2\sqrt{k}/(1+k)$:

$$L(k(u)) = \frac{4\pi(AB-1)^{1/2}}{B^{1/2}k^{1/2}K} \sum_{0}^{\infty} \frac{(-1)^m q_0^{m+1/2}}{1+q_0^{2m+1}} \sin\left(m+\frac{1}{2}\right)\frac{\pi u}{2K} \tag{B.17}$$

which is readily identifiable[26] in terms of elliptic functions of modulus k_0 as

$$L(k(u)) = 4\left(\frac{AB-1}{B}\right)^{1/2} \frac{1-k}{1+k} \operatorname{sd}((1+k)u/2 \mid k_0) \tag{B.18}$$

The solution is expressed in a more transparent from by finding a function of z with the same zeros, poles, and periodicity factors when translated to u. Such a function is

$$c\frac{(z+1)}{[(z-B)(z-B^{-1})]^{1/2}} \tag{B.19}$$

which is $4K$-antiperiodic and $4iK'$-periodic. The constant c is fixed by considering the behavior at $u=2K$, i.e., $z=1$. This is the first method of solution which we found for (3.8). There is an analogous method for (3.9).

APPENDIX C. HAMILTONIAN LIMIT

We explore the connection between our work and the Hamiltonian-limit work of Barber *et al.*[3] First note that in ref. 3, K_2 is specified as the coupling in the transfer direction. Retaining our labeling and letting $K_1 \to \infty$ and $K_2 \to 0$ so that $\lambda = K_2/K_1^*$ is fixed, we see that $\lambda = B$ and $A \to \infty$ in the formulas of this paper in order to capture the Hamiltonian limit. Referring to (2.5), (2.6), and (2.10) we find that, asymptotically,

$$V' \approx \exp\left(-2K_1^* \sum_k \varepsilon(k)(\bar{X}_k^\dagger \bar{X}_k - \tfrac{1}{2})\right) \tag{C.1}$$

where

$$\varepsilon(k) = (1+\lambda^2 - 2\lambda\cos k)^{1/2} \tag{C.2}$$

and the $\bar{X}_k$ are obtained from (2.6) by taking the Hamiltonian limit on (2.14) and (2.15). Note that in (C.2), our wavenumber is shifted from that of ref. 3 by π. Using the Hamiltonian limit of (3.6) allows a connection to be made to the function F of ref. 3 [see their Eqs. (6.11), (6.20), and the following line; also see their (6.22)]. Taking our (3.11) and pairing k with $-k$ in the integral, and noting the effect on $K(e^{ik})$ of reversing k expressed in the paragraph between (4.3) and (4.4), gives Eq. (6.21) of ref. 3 after some algebra. In ref. 3 the authors did not write down the Hamiltonian-limit analog of our (4.2) and were therefore not led to the analog of our (4.1), which is the key to the Wiener–Hopf treatment, nor did they notice the relationship to the Yang singular integral operator.

Finally, the key equation (6.12) in ref. 3 is the same starting point as ours, based on the technique adumbrated in refs. 19 and 20.

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